

Quantum Error Correction of Time-Correlated Errors

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Abstract

The complexity of the error correction circuitry forces us to design quantum error correction codes capable of correcting a single error per error correction cycle. Yet, time-correlated error are common for physical implementations of quantum systems; an error corrected during the previous cycle may reoccur later due to physical processes specific for each physical implementation of the qubits. In this paper we study quantum error correction for a restricted class of time-correlated errors in a spin-boson model. The algorithm we propose allows the correction of two errors per error correction cycle, provided that one of them is time-correlated. The algorithm can be applied to any stabilizer code when the two logical qubits $|0_L\rangle$ and $|1_L\rangle$ are entangled states of 2^n basis states in \mathcal{H}_{2^n} .

1 Quantum Error Correction

Quantum states are subject to decoherence and the question whether a reliable quantum computer could be built was asked early on. A “pure state” $|\varphi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ may be transformed as a result of the interaction with the environment into a “mixed state” with density matrix:

$$\rho = |\alpha_0|^2 |0\rangle\langle 0| + |\alpha_1|^2 |1\rangle\langle 1|.$$

Other forms of decoherence, e.g. leakage may affect the state probability amplitude as well.

The initial thought was that a quantum computation could only be carried out successfully if its duration is shorter than the decoherence time of the quantum computer. As we shall see in Section 2, the decoherence time ranges from about 10^4 seconds for the nuclear spin embodiment of a qubit, to 10^{-9} seconds for quantum dots based upon charge. Thus, it seemed very problematic that a quantum computer could be built unless we have a mechanism to deal periodically with errors. Now we know [16] that quantum error correcting codes can be used to ensure fault-tolerant quantum computing; quantum error correction allows us to deal algorithmically with decoherence. There is a significant price to pay to achieve fault-tolerance through error correction: the number of qubits required to correct errors could be several orders of magnitude larger than the number of “useful” qubits [6].

We describe the effect of the environment upon a qubit as a transformation given by Pauli operators: (i) the state of the qubit is unchanged if we apply the σ_I operator; (ii) a bit-flip error is the result of applying the transformation given by σ_x ; (iii) a phase-flip error is the result of applying the transformation given by σ_z ; and (iv) a bit-and-phase flip error is the result of applying the transformation given by $\sigma_y = i\sigma_x\sigma_z$.

A quantum error correcting scheme takes advantage of entanglement in two ways:

- We entangle one qubit carrying information with $(n-1)$ other qubits initially in state $|0\rangle$ and create an n -qubit quantum codeword which is more resilient to errors.

- We entangle the n qubits of the quantum codeword with ancilla qubits in such a way that we can measure the ancilla qubits to determine the error syndrome without altering the state of the n -qubit codeword, by performing a so called non demolition measurement. The error syndrome tells if the individual qubits of the codeword have been affected by errors as well as the type of error.
- Finally, we correct the error(s).

Even though the no-cloning theorem prohibits the replication of a quantum state, we are able to encode a single logical qubit as multiple physical qubits and thus we can correct quantum errors [18]. For example, we can encode the state of a qubit

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

as a linear combination of $|00000\rangle$ and $|11111\rangle$:

$$|\varphi\rangle = \alpha_0 |00000\rangle + \alpha_1 |11111\rangle.$$

Alternatively, we can encode the qubit in state $|\psi\rangle$ as:

$$|\varphi\rangle = \alpha_0 |0_L\rangle + \alpha_1 |1_L\rangle,$$

with $|0_L\rangle$ and $|1_L\rangle$ expressed as a superposition of codewords of a classical linear code. In this case all codewords are superpositions of vectors in \mathcal{H}_{2^5} , a Hilbert space of dimension 2^5 . Steane's seven-qubit code [17] and Shor's nine qubit code [3] are based upon this scheme.

When we use the first encoding scheme, a random error can cause departures from the subspace spanned by $|00000\rangle$ and $|11111\rangle$. We should be able to correct small bit-flip errors because the component which was $|00000\rangle$ is likely to remain in a sub-space $\mathcal{C}_0 \subset \mathcal{H}_{2^5}$ spanned by the six vectors:

$$|00000\rangle, |00001\rangle, |00010\rangle, |00100\rangle, |01000\rangle, |10000\rangle$$

while the component which was $|11111\rangle$ is likely to remain in a sub-space $\mathcal{C}_1 \subset \mathcal{H}_{2^5}$ spanned by the six vectors,

$$|11111\rangle, |11110\rangle, |11101\rangle, |11011\rangle, |10111\rangle, |01111\rangle.$$

The two subspaces are disjoint:

$$\mathcal{C}_0 \cap \mathcal{C}_1 = \emptyset$$

thus we are able to correct a bit-flip error of any single physical qubit. This procedure is reminiscent of the basic idea of classical error correction when we determine the Hamming sphere an n -tuple belongs to, and then correct it as the codeword at the center of the sphere.

A quantum code encodes one logical qubit into n physical qubits in the Hilbert space \mathcal{H}_{2^n} with basis vectors $\{|0\rangle, |1\rangle, \dots, |i\rangle, \dots, |2^n - 1\rangle\}$. The two logical qubits are entangled states in \mathcal{H}_{2^n} :

$$|0_L\rangle = \sum_i \alpha_i |i\rangle \quad \text{and} \quad |1_L\rangle = \sum_i \beta_i |i\rangle$$

A quantum error correcting code must map coherently the two-dimensional Hilbert space spanned by $|0_L\rangle$ and $|1_L\rangle$ into two-dimensional Hilbert spaces corresponding to bit-flip, phase-flip, as well as bit-and-phase flip of each of the n qubits to ensure that the code is capable of correcting the three types of error for each qubit. The *quantum Hamming bound*:

$$2(3n + 1) \leq 2^n.$$

established by Laflamme, Miquel, Paz, and Zurek in [11] allows us to say that $n = 5$ is the smallest number of qubits required to encode the two superposition states $|0_L\rangle$ and $|1_L\rangle$, and then be able to recover them regardless of the qubit in error and the type of errors.

The same paper [11] introduces a family of 5-qubit quantum error correcting codes. The code \mathcal{Q} with the logical codewords

$$\begin{aligned} |0_L\rangle &= \frac{1}{4} (|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\ &\quad - |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle) \\ |1_L\rangle &= \frac{1}{4} (|11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\ &\quad - |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle) \end{aligned}$$

is a member of this family. The code \mathcal{Q} is a *perfect quantum error correcting code* because a logical codeword consists of the smallest number n of qubits to satisfy the inequality from [11]. Recall that a *perfect linear code* $[n, k, d]$ with $d = 2e + 1$ is one where Hamming spheres of radius e are all disjoint and exhaust the entire space of n -tuples.

Different physical implementations reveal that the interactions of the qubits with the environment are more complex and force us to consider spatially- as well as, time-correlated errors. If the qubits on an n qubit register are confined to a 3D structure, an error affecting one qubit will propagate to the qubits in a volume centered around the qubit in error. Spatially-correlated errors and means to deal with the spatial noise are analyzed in [1, 4, 9]. An error affecting qubit i of an n -qubit register may affect other qubits of the register. An error affecting qubit i at time t and corrected at time $t + \Delta$ may have further effect either on qubit i or on other qubits of the register. The error model considered in this paper is based upon a recent study which addresses the problem of reliable quantum computing using solid state systems [13].

There are two obvious approaches to deal with quantum correlated errors :

- (i) design a code capable to correct these two or more errors, or
- (ii) use the classical information regarding past errors and quantum error correcting codes capable to correct a single error.

When a quantum error correcting code uses a large number of qubits and quantum gates it becomes increasingly more difficult to carry out the encoding and syndrome measurements during a single quantum error correction cycle. For example, if we encode a logical qubit using Shor's nine qubit code we need 9 physical qubits. If we use a two-level convolutional code based upon Shor's code then we need 81 physical qubits and $63 = 7 \times 9$ ancilla qubits to ensure that the circuit for syndrome calculation is fault-tolerant. While constructing quantum codes capable of correcting a larger number of errors is possible, we believe that the price to pay, the increase in circuit complexity makes this solution undesirable; this motivates our interest in the second approach.

2 Time-Correlated Quantum Errors

Figure 1 illustrates the evolution in time of two qubits, i and j for two error correction cycles. The first error correction cycle ends at time t_2 and the second at time t_5 . At time t_1 qubit i is affected by decoherence and flips; at time t_2 it is flipped back to its original state during the first error correction step; at time t_3 qubit j is affected by decoherence and it is flipped; at time t_4 the correlation effect discussed in this section affects qubit i and flips it to an error state. If an algorithm is capable to correct one correlated error in addition to a "new" error, then during the second error correction the errors affecting qubits i and j are corrected at time t_5 .

The quantum computer and the environment are entangled during the quantum computation. When we measure the state of the quantum computer this entanglement is translated into a probability ϵ that the measured state differs from the expected one. This probability of error determines the actual number of errors a quantum code is expected to correct.

If τ_{gate} is the time required for a single gate operation and τ_{dch} is the decoherence time of a qubit, then n_{gates} , the number of gates that can be traversed by a register before it is affected by decoherence is given by:

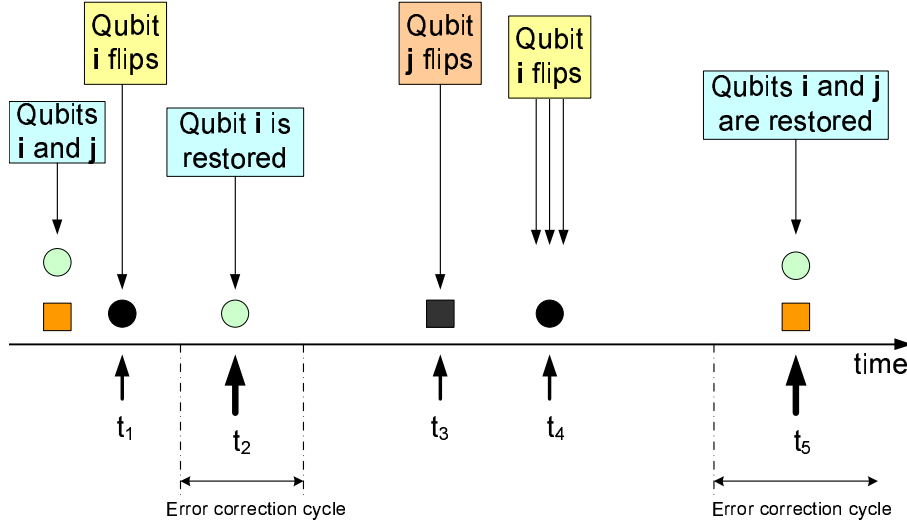


Figure 1: Time-correlated quantum errors. Two consecutive error correction cycles occurring at time t_2 and t_5 are shown. At time t_5 a code designed to correct a single error will fail while a code capable to handle time-correlated errors will correct the “new” error of qubit j and the “old” correlated error of qubit i .

Table 1: The time required for a single gate operation, τ_{gate} , the decoherence time of a qubit, τ_{dch} , and the number of gates that can be traversed before a register of qubits is affected by decoherence, n_{gates} .

Qubit implementation	$\tau_{dch}(sec)$	$\tau_{gate}(sec)$	n_{gates}
Nuclear spin	10^4	10^{-3}	10^7
Trapped Indium ion	10^{-1}	10^{-14}	10^{13}
Quantum dots/charge	10^{-9}	10^{-12}	10^3
Quantum dots/spin	10^{-6}	10^{-9}	10^3
Optical cavity	10^{-5}	10^{-14}	10^9

$$n_{gates} = \frac{\tau_{dch}}{\tau_{gate}}.$$

Quantum error correction is intimately related to the physical processes which cause decoherence. Table 1 presents sample values of the time required for a single gate operation τ_{gate} , the decoherence time of a qubit, τ_{dch} , and n_{gates} , the number of gates that can be traversed before a register of qubits is affected by decoherence, for several qubit implementations [8, 12, 15, 19]. We notice a fair range of values for the number of quantum gate operations that can be performed before decoherence affects the state.

The information in Table 1, in particular the decoherence time, can be used to determine the length of an error correction cycle, the time elapsed between two consecutive error correction steps. The number of quantum gate operations limits the complexity of the quantum circuit required for quantum error correction.

The Quantum Error Correction theory is based upon the assumption that the quantum system has a constant error rate ϵ . This implies that once we correct an error at time t_c , the system behavior at time $t > t_c$ is decoupled from events prior to t_c .

The Markovian error model is not consistent with some physical implementations of qubits; for example, in a recent paper Novais and Baranger [13] discuss the decoherence in a spin-boson model

which is applicable, for instance, to quantum dots. The authors assume that the qubits are perfect, thus eventual errors are due to dephasing and consider a linear coupling of the qubits to an ohmic bath. Using this model the authors analyze the three-qubit Steane's code. They calculate the probability of having errors in quantum error correction cycles starting at times t_1 and t_2 and show that the probability of errors consists of two terms; the first is the uncorrelated probability and the second is the contribution due to correlation between errors in different cycles (Δ is the period of the error correcting cycle):

$$P \approx \left(\frac{\epsilon}{2}\right)^2 + \frac{\lambda^4 \Delta^4}{8(t_1 - t_2)^4}$$

We see that correlations in the quantum system decay algebraically in time, and the latest error will re-influence the system with a much higher probability than others.

Only phase-flip errors are discussed in detail in [13]. The approach introduced by the authors is very general and can be extended to include other types of errors. Nevertheless, for other physical models, one does expect major departures from the results presented in [13].

3 Stabilizer Codes

The stabilizer formalism is a succinct manner of describing a quantum error correcting code by a set of quantum operators [7]. We first review several concepts and properties of stabilizer codes.

The 1-qubit Pauli group \mathcal{G}_1 consists of the Pauli operators, σ_I , σ_x , σ_y , and σ_z together with the multiplicative factors ± 1 and $\pm i$:

$$\mathcal{G}_1 \equiv \{\pm\sigma_I, \pm i\sigma_I, \pm\sigma_x, \pm i\sigma_x, \pm\sigma_y, \pm i\sigma_y, \pm\sigma_z, \pm i\sigma_z\}.$$

The generators of \mathcal{G}_1 are:

$$\langle\sigma_x, \sigma_z, i\sigma_I\rangle.$$

The n -qubit Pauli group \mathcal{G}_n consists of the 4^n tensor products of σ_I , σ_x , σ_y , and σ_z with an overall phase of ± 1 or $\pm i$. Elements of the group can be used to describe the error operators applied to an n -qubit register. The *weight* of such an operator in \mathcal{G}_n is equal to the number of tensor factors which are not equal to σ_I . The stabilizer S of code \mathcal{Q} is a subgroup of the n -qubit Pauli group, $S \subset \mathcal{G}_n$. The generators of the subgroup S are:

$$M = \{\mathbf{M}_1, \mathbf{M}_2 \dots \mathbf{M}_q\}.$$

The eigenvectors of the generators $\{\mathbf{M}_1, \mathbf{M}_2 \dots \mathbf{M}_q\}$ have special properties: those corresponding to eigenvalues of $+1$ are the codewords of \mathcal{Q} and those corresponding to eigenvalues of -1 are codewords affected by errors. If a vector $|\psi_i\rangle \in \mathcal{H}_n$ satisfies,

$$\mathbf{M}_j |\psi_i\rangle = (+1) |\psi_i\rangle \quad \forall \mathbf{M}_j \in M$$

then $|\psi_i\rangle$ is a codeword, $|\psi_i\rangle \in \mathcal{Q}$. This justifies the name given to the set S , any operator in S stabilizes a codeword, leaving the state of a codeword unchanged. On the other hand if:

$$\mathbf{M}_j |\varphi_k\rangle = (-1) |\varphi_k\rangle.$$

then $|\varphi_k\rangle = \mathbf{E}_i |\psi_k\rangle$, the state $|\varphi_k\rangle$ is a codeword $|\psi_k\rangle \in \mathcal{Q}$ affected by error \mathbf{E}_i . The error operators affecting codewords in \mathcal{Q} , $E = \{\mathbf{E}_1, \mathbf{E}_2 \dots\}$, are also a subgroup of the n -qubit Pauli group

$$E \subset \mathcal{G}_n.$$

Each error operator \mathbf{E}_i is a tensor product of n Pauli matrices. Its weight is equal to the number of errors affecting a quantum work, thus the number of Pauli matrices other than σ_I .

The coding space:

$$Q = \{ |\psi_i\rangle \in \mathcal{H}_n \text{ such that } M_j |\psi_i\rangle = (+1) |\psi_i\rangle, \quad \forall M_j \in S \}$$

is the space of all vectors $|\psi_i\rangle$ fixed by S

It is easy to prove that S is a stabilizer of a non-trivial Hilbert subspace $V_{2^n} \subset \mathcal{H}_{2^n}$ if and only if:

1. $S = \{S_1, S_2, \dots\}$ is an Abelian group:

$$S_i S_j = S_j S_i, \quad \forall S_i, S_j \in S \quad i \neq j.$$

2. The identity matrix multiplied by -1 is not in S :

$$-(\sigma_I^{\otimes n}) \notin S.$$

If \mathbf{E} is an error operator, and \mathbf{E} anti-commutes with some element $\mathbf{M} \in S$, then \mathbf{E} can be detected, since for any $|\psi_i\rangle \in Q$:

$$\mathbf{M}\mathbf{E} |\psi_i\rangle = -\mathbf{E}\mathbf{M} |\psi_i\rangle = -\mathbf{E} |\psi_i\rangle$$

Definition. The normalizer $N(S)$ consists of operators $\mathbf{E} \in \mathcal{G}_n$ such that $\mathbf{E}S_i\mathbf{E}^\dagger \in S, \forall S_i \in S$. The distance d of a stabilizer code is the minimum weight of an element in $N(S) - S$ with $N(S)$ the normalizer of S . An $[n, k, d]$ stabilizer code is an $[n, k]$ code stabilized by S and with distance d ; n is the length of a codeword, and k the number of information symbols.

An $[n, k, d]$ stabilizer code with the minimum distance $d = 2e + 1$ can correct at most e arbitrary quantum errors or $2e$ errors whose location is well known. In [5] this property of a stabilizer code is used for syndrome decoding for optical cluster state quantum computation.

Given an $[n, k, d]$ stabilizer code the cardinalities of the stabilizer S and of its generator M are:

$$|S| = 2^{n-k}, \quad |M| = n - k$$

The error syndrome corresponding to the stabilizer \mathbf{M}_j is a function of the error operator, \mathbf{E} , defined as:

$$f_{\mathbf{M}_j}(\mathbf{E}) : \mathcal{G} \mapsto \mathbf{Z}_2 \quad f_{\mathbf{M}_j}(\mathbf{E}) = \begin{cases} 0 & \text{if } [\mathbf{M}_j, \mathbf{E}] = 0 \\ 1 & \text{if } \{\mathbf{M}_j, \mathbf{E}\} = 0, \end{cases}$$

where $[\mathbf{M}_j, \mathbf{E}]$ is the commutator and $\{\mathbf{M}_j, \mathbf{E}\}$ the anti-commutator of operators \mathbf{M}_j and \mathbf{E} . Let $f(\mathbf{E})$ be the $(n - k)$ -bit integer given by the binary vector:

$$f(\mathbf{E}) = (f_{\mathbf{M}_1}(\mathbf{E}) f_{\mathbf{M}_2}(\mathbf{E}) \dots f_{\mathbf{M}_{n-k}}(\mathbf{E})).$$

This $(n - k)$ -bit integer is called the *syndrome of error* \mathbf{E} .

Proposition. The error syndrome uniquely identifies the qubit(s) in error if and only if the subsets of the stabilizer group which anti-commute with the error operators are distinct.

An error can be identified and corrected only if it can be distinguished from any other error in the error set. Let $Q(S)$ be the stabilizer code with stabilizer S . The *Correctable Set of Errors* for $Q(S)$ includes all errors which can be detected by S and have distinct error syndromes.

Table 2: Single error operators for the 5-qubit code and the generator(s) that anti-commute with each operator. Subscripts indicate positions of errors, e.g. X_3 means a bit-flip error on the third qubit, $X_3 = \sigma_I \otimes \sigma_I \otimes \sigma_x \otimes \sigma_I \otimes \sigma_I$.

Error operator	Generator(s)	Error operator	Generator(s)	Error operator	Generator(s)
\mathbf{X}_1	\mathbf{M}_4	\mathbf{Z}_1	$\mathbf{M}_1, \mathbf{M}_3$	\mathbf{Y}_1	$\mathbf{M}_1, \mathbf{M}_3, \mathbf{M}_4$
\mathbf{X}_2	\mathbf{M}_1	\mathbf{Z}_2	$\mathbf{M}_2, \mathbf{M}_4$	\mathbf{Y}_2	$\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_4$
\mathbf{X}_3	$\mathbf{M}_1, \mathbf{M}_2$	\mathbf{Z}_3	\mathbf{M}_3	\mathbf{Y}_3	$\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$
\mathbf{X}_4	$\mathbf{M}_2, \mathbf{M}_3$	\mathbf{Z}_4	$\mathbf{M}_1, \mathbf{M}_4$	\mathbf{Y}_4	$\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$
\mathbf{X}_5	$\mathbf{M}_3, \mathbf{M}_4$	\mathbf{Z}_5	\mathbf{M}_2	\mathbf{Y}_5	$\mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4$

Corollary. *Given a quantum error correcting code Q capable to correct e_u errors, the syndrome does not allow us to distinguish the case when more than e_u qubits are in error. If we can identify the e_c correlated errors in system, the code is capable of correcting these $e_u + e_c$ errors.*

Proof: Assume that F_1, F_2 cause at most e_u qubits to be in error, thus F_1, F_2 are included in the *Correctable Set of Errors* of Q . Assuming errors F_1 and F_2 are distinguishable, there must exist some operator $M \in S$ which commutes with one of them, and anti-commutes with the other:

$$F_1^T F_2 M = -M F_1^T F_2$$

If we know the exact correlated errors E in the system, then:

$$(E^T F_1)^T (E^T F_2) M = (F_1^T E E^T F_2) M = F_1^T F_2 M = -M F_1^T F_2 = -M (E^T F_1)^T (E^T F_2)$$

which means that the stabilizer M commutes with one of the two errors $E^T F_1, E^T F_2$ and anti-commutes with the other. So error $E^T F_1$ is distinguishable from error $E^T F_2$. Therefore, if we know the exact prior errors E , we can identify and correct any $E^T F_i$ errors with the weight of F_i equal or less than e_u .

For example, consider a 5-qubit quantum error-correcting code Q with $n = 5$ and $k = 1$ from [11] discussed in Section 1. The stabilizer S of this code is described by a group of 4 generators:

$$M = \{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4\} \quad \text{with the generators:}$$

$$\begin{aligned} \mathbf{M}_1 &= \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x \otimes \sigma_I, & \mathbf{M}_2 &= \sigma_I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x, \\ \mathbf{M}_3 &= \sigma_x \otimes \sigma_I \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z, & \mathbf{M}_4 &= \sigma_z \otimes \sigma_x \otimes \sigma_I \otimes \sigma_x \otimes \sigma_z. \end{aligned}$$

It is easy to see that two codewords are eigenvectors of the stabilizers with an eigenvalue of $(+1)$:

$$\mathbf{M}_j |0_L\rangle = (+1) |0_L\rangle \quad \text{and} \quad \mathbf{M}_j |1_L\rangle = (+1) |1_L\rangle, \quad 1 \leq j \leq 4.$$

Note also that M is an Abelian subgroup, each generator commutes with all the others.

Table 2 lists single error operators and the generator(s) which anti-commute with each operator. For example, \mathbf{X}_1 anti-commutes with \mathbf{M}_4 , thus a bit-flip on the first qubit can be detected; \mathbf{Z}_1 anti-commutes with \mathbf{M}_1 and \mathbf{M}_3 , thus a phase-flip of the first qubit can also be detected. Since each of these 15 errors anti-commute with distinct subsets of S we can distinguish individual errors and then correct them. An example shows that the code cannot detect two qubit errors; indeed, the two bit-flip error

$$\mathbf{X}_1 \mathbf{X}_2 = \sigma_x \otimes \sigma_x \otimes \sigma_I \otimes \sigma_I \otimes \sigma_I$$

is indistinguishable from $\mathbf{Z}_4 = \sigma_I \otimes \sigma_I \otimes \sigma_I \otimes \sigma_z \otimes \sigma_I$ because both $\mathbf{X}_1 \mathbf{X}_2$ and \mathbf{Z}_4 anti-commute with the same subset of stabilizers, $\{\mathbf{M}_1, \mathbf{M}_4\}$, and give the same error syndrome. Therefore, *the 5-qubit code can correct any single qubit error, but cannot correct two qubit errors.*

If we know the correlated error in the system, for example, \mathbf{X}_3 , then, from Table 2, it is easy to see that the errors $\mathbf{X}_3\mathbf{E}_i$ have distinct error syndromes for $1 \leq i \leq 15$. Therefore we can identify these errors and correct them.

4 Time-Correlated Errors and Stabilizer Codes

Classical, as well as quantum error correction schemes allow us to construct codes with a well-defined error correction capability. If a code is designed to correct e errors, it will fail whenever more than e errors occur.

From previous section we know that if time-correlated errors are present in the system, we can not always detect them through the calculation of the syndrome. The same syndrome could signal the presence of a single error, two, or more errors, as shown by our example in Section 3; we can remove this ambiguity when there are two errors and one of them is a time-correlated error.

In this section we extend the error correction capabilities of any stabilizer code designed to correct a single error (bit-flip, phase-flip error, or bit-and-phase flip) and allow the code to correct an additional time-correlated error. The standard assumptions for quantum error correction are:

- Quantum gates are perfect and operate much faster than the characteristic response of the environment;
- The states of the computer can be prepared with no errors.

We also assume that:

- There is no spatial-correlation of errors, a qubit in error does not influence its neighbors;
- In each error correcting cycle, in addition to a new error E_a that occurs with a constant probability ε_a , a time-correlated error E_b may occur with probability $\varepsilon_b(t)$. As correlations decay in time, the qubit affected by error during the previous error correction cycle, has the highest probability to relapse.

Quantum error correction requires *non-demolition measurements of the error syndrome* in order to preserve the state of the physical qubits. In other words, a measurement of the probe (the ancilla qubits) should not influence the free motion of the signal system. The syndrome has to identify precisely the qubit(s) in error and the type of error(s). Thus, the qubits of the syndrome are either in the state with density $|0\rangle\langle 0|$ or in the state with density $|1\rangle\langle 1|$, which represent classical information.

A quantum non-demolition measurement allows us to construct the error syndrome $\Sigma_{current}$. After examining the syndrome $\Sigma_{current}$ an error correcting algorithm should be able to decide whether:

1. No error has occurred; no action should be taken;
2. One “new” error, E_a , has occurred; then we apply the corresponding Pauli transformation to the qubit in error;
3. Two or more errors have occurred. There are two distinct possibilities: (a) We have a “new” error as well as an “old” one, the time-correlated error; (b) There are two or more “new” errors. A quantum error correcting code capable of correcting a single error will fail in both cases.

It is rather hard to distinguish the last two possibilities. For perfect codes, the syndrome S_{ab} corresponding to two errors, E_a and E_b , is always identical to the syndrome S_c for some single error E_c . Thus, for perfect quantum codes the stabilizer formalism does not allow us to distinguish two errors from a single one; for a non-perfect code it is sometimes possible to distinguish the two

syndromes and then using the knowledge regarding the time-correlated error it may be possible to correct both the “old” and the “new” error.

We now describe an algorithm based on the stabilizer formalism capable to handle case 3(a) for perfect as well as non-perfect quantum codes. We assume that the quantum code has a codeword consisting of n qubits and uses k ancilla qubits for syndrome measurement.

The outline of our algorithm:

1. At the end of an error correction cycle entangle the qubit affected by the error with two additional ancilla qubits. Thus, if the qubit relapses, the time-correlated error will propagate to the two additional ancilla qubits.
2. Carry out an *extended syndrome measurement* of the codeword together with the two additional ancilla qubits. This syndrome measurement should not alter the state of the codeword and keep the entanglement between the codeword and the two additional ancilla qubits intact.
3. Disentangle the two additional ancilla qubits from the codeword and then measure the two additional ancilla qubits.
4. Carry out the error correction according to the outcomes of Steps 2 and 3.

Next we use the Steane code to illustrate the details of the algorithm. The generators of the Steane 7-qubit code are:

$$\begin{aligned} \mathbf{M}_1 &= \sigma_I \otimes \sigma_I \otimes \sigma_I \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x, & \mathbf{M}_2 &= \sigma_I \otimes \sigma_x \otimes \sigma_x \otimes \sigma_I \otimes \sigma_I \otimes \sigma_x \otimes \sigma_x, \\ \mathbf{M}_3 &= \sigma_x \otimes \sigma_I \otimes \sigma_x \otimes \sigma_I \otimes \sigma_x \otimes \sigma_I \otimes \sigma_x, & \mathbf{M}_4 &= \sigma_I \otimes \sigma_I \otimes \sigma_I \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z, \\ \mathbf{M}_5 &= \sigma_I \otimes \sigma_z \otimes \sigma_z \otimes \sigma_I \otimes \sigma_I \otimes \sigma_z \otimes \sigma_z, & \mathbf{M}_6 &= \sigma_z \otimes \sigma_I \otimes \sigma_z \otimes \sigma_I \otimes \sigma_z \otimes \sigma_I \otimes \sigma_z. \end{aligned}$$

1. Entangle the two additional ancilla qubits with the original code word. The time-correlated error may reoccur in different physical systems differently: a bit-flip could lead to a bit-flip, a phase-flip, or a bit-phase-flip. Therefore, we need to “backup” the qubit corrected during the previous error correction cycle in both \mathbf{X} and \mathbf{Z} basis. We entangle the qubit affected by an error with two additional ancilla qubits: a CNOT gate will duplicate the qubit in \mathbf{Z} basis using the first ancilla qubit, the second ancilla will duplicate the qubit in \mathbf{X} basis using two Hadamard gates and a CNOT gate (called an HCNOT gate), see Figure 2(a). In this example, qubit 3 is the one affected by error in the last error correction cycle.

2. Extended syndrome measurement. Measure the extended syndrome S_{n+2} for an $(n+2)$ extended codeword consisting of the original codeword and the two extra ancilla qubit. Call Σ the result of the measurement of the extended syndrome S_{n+2} . When implementing this extended syndrome measurement, we need consider two aspects: (i) the entanglement between the additional ancilla and codewords should not be disturbed, since we need to disentangle the extra ancilla from the code and keep the codeword intact; and (ii) the new stabilizers (or generators) should also stabilize the codeword and satisfy all the stabilizers’ requirements. In our approach, the additional ancilla qubit \mathbf{A} is a copy of the control qubit in \mathbf{Z} basis. To keep the entanglement, we copy all \mathbf{X} operations on the control qubit to qubit \mathbf{A} . Similarly, the additional ancilla qubit \mathbf{B} is a copy of the control qubit in \mathbf{X} basis. We replicate all \mathbf{Z} operations performed on the control qubit to this qubit. Since it is in \mathbf{X} basis, the \mathbf{Z} operations will change into \mathbf{X} operations. Consider the example of the Steane code and the time-correlated error on qubit 3, Figure 2(b). In this case we replicate all \mathbf{X} operations performed on qubit 3 to ancilla qubit \mathbf{A} , and replicate all \mathbf{Z} operations to additional ancilla qubit \mathbf{B} as \mathbf{X} operations. These operations will keep the entanglement intact during the syndrome measurement process. Also, it is easy to check that the new stabilizers (i) commute with each other, and (ii) stabilize the codewords.

3. Disentangle and measure the two additional ancilla qubits. After the syndrome measurement, we use a circuit similar to the one in Figure 2(a) to disentangle the two additional ancilla

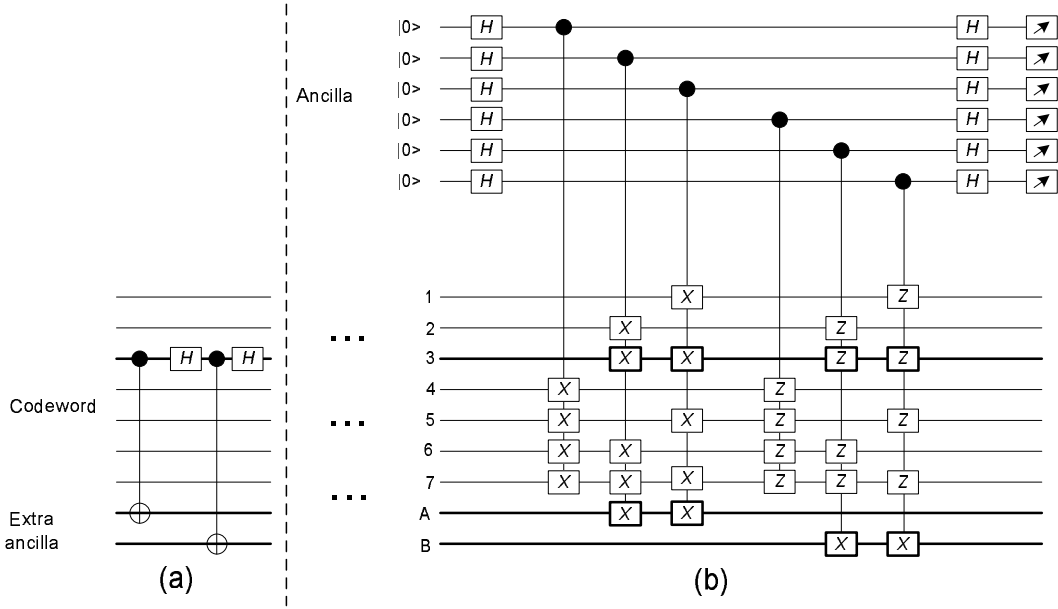


Figure 2: (a) Duplicate the qubit affected error during the last cycle in both \mathbf{X} and \mathbf{Z} basis; (b) Extended syndrome measurement on both the codeword and the additional ancilla qubits.

qubits from the codeword. The gates in this circuit are in reverse order as in Figure 2(a). Since the entanglement is not affected during the extended syndrome measurement, this process will disentangle the extra ancilla from the codewords and keep the codeword in its original state. If qubit 3 is affected by the time-correlated error, bit-flip, phase-flip or both, the disentanglement circuit will propagate these errors to the extra ancilla qubits: a \mathbf{CNOT} gate will propagate a bit-flip of the control qubit to the target, and an \mathbf{HCNOT} gate will propagate a phase-flip of the control qubit to the target qubit as a bit-flip, Figure 3. Therefore, measuring the additional ancilla qubits after the disentanglement will give us the information regarding the type of error qubit 3 has released to.

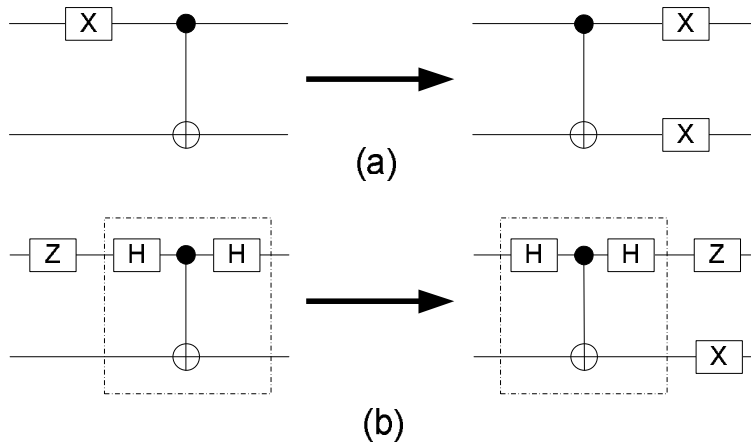


Figure 3: (a) A \mathbf{CNOT} gate propagates a bit-flip of the control qubit to the target qubit (b) An \mathbf{HCNOT} gate propagates a phase-flip on the control qubit to the target qubit as a bit-flip.

4. Correct the errors according to the output of the extended syndrome measurement and of the measurement of the two additional ancilla qubits. Consider the following

scenarios:

1. A “new” error affects a qubit i of the codeword, while qubit $j, j \neq i$ was the one in error during the previous cycle. Then the measurement of the extended syndrome produces the same outcome as the measurement of the original syndrome and the new error is identified.
2. The time-correlated error of qubit j occurs and there is no “new” error. The error propagates to the two ancilla qubits. The measurement of the original syndrome allows us to identify the single error.
3. There are two errors, the time-correlated error of qubit j and a “new” error affecting qubits $i, i \neq j$ of the codeword. Then the extended syndrome measurement gives the combination of these two errors: $\Sigma = \Sigma_j \oplus \Sigma_i$. The measurement of the two additional ancilla qubits reveals the type of the time-correlated error: 10 – bit-flip, 01 – phase-flip, 11 – bit-and-phase-flip, see Figure 3. Knowing that qubit j is in error and the corresponding error type we use the syndrome table to find the new error on qubit i .
4. There is a single “new” error affecting one of the two extra ancilla qubits. This error may propagate to the qubit of the codeword the ancilla is entangled with. We treat this scenario as (1). For example, in Figure 2, a phase-flip of the ancilla qubit **A** propagates to qubit 3. The measurement of the original syndrome allows us to identify this error on qubit 3.
5. There are two errors, the time-correlated error on qubit j of the codeword and an error on the additional ancilla entangled with qubit j . As we mention in scenario (3), the error on extra ancilla may back propagate to qubit j . The net result is that in the codeword, only the qubit j is affected by error: bit-flip, phase-flip or bit-and-phase-flip. This error is indicated by the syndrome and can be corrected as in scenario (1).

The error correction process can be summarized as follows:

1. If the syndrome Σ indicates an error on qubit j , the one affected by error during the last cycle (qubit 3 in our example), correct the error;
2. Else, determine the outcome of a measurement of the two additional ancilla qubits:
 - (a) If the outcome is 00 , and Σ corresponds to a single error syndrome, correct this error;
 - (b) If the outcome is either 01 , or 10 , or 11 , qubit j in error during the previous cycle has relapsed. Then the syndrome is $\Sigma_i = \Sigma \otimes \Sigma_j$. Knowing that qubit j is in error and the corresponding error type (10 – bit-flip, 01 – phase-flip, 11 – bit-and-phase-flip) we use the syndrome table to find the new error affecting qubit i .
 - (c) Otherwise, there are two or more new errors in the system and they cannot be corrected.

Example. Consider again the Steane seven-qubit code. Table 3 shows the syndromes for all single errors for the Steane 7-qubit quantum code. Assume that during the last error correction cycle qubit $j = 3$ was affected by an error. We use a CNOT gate to entangle qubit 3 with the additional ancilla qubit **A** in **Z** basis, and an HCNOT gate to entangle it with qubit **B** in **X** basis, Figure 2.

1. $\Sigma = 011000$, indicates a phase-flip on qubit 3. We correct this error. In this case the measurement of the additional ancilla qubit could be non-zero, but this will not affect the codeword.
2. $\Sigma = 110000$ and the outcome of the measurement of the two additional ancilla qubits is 00 . In this case there is only one “new” error in the system, a phase-flip on qubit $i = 6$ as indicated by Σ . We correct this error.
3. $\Sigma = 110000$ and the outcome of the measurement of the additional ancilla qubits is 01 . In this case $\Sigma_j = \Sigma_{\text{phase-flip}} = \Sigma_{Z3} = 011000$. Therefore, the new error has syndrome $\Sigma_i = 110000 \otimes 011000 = 101000$; this indicates a phase-flip of qubit $i = 5$. Correct the phase-flips on qubit 3 and qubit 5.

Table 3: The syndromes for each of the three types of errors of each qubit of a codeword for the Steane 7-qubit code: X_{1-7} bit-flip, Z_{1-7} phase-flip, and Y_{1-7} bit-and-phase flip.

Error	Syndrome	Error	Syndrome	Error	Syndrome
X_1	000001	Z_1	001000	Y_1	001001
X_2	000010	Z_2	010000	Y_2	010010
X_3	000011	Z_3	011000	Y_3	011011
X_4	000100	Z_4	100000	Y_4	100100
X_5	000101	Z_5	101000	Y_5	101101
X_6	000110	Z_6	110000	Y_6	110110
X_7	000111	Z_7	111000	Y_7	111111

The circuits described in our paper have been tested using the Stabilizer Circuit Simulator [2]; sample codes are provided in the Appendix. One can test different error scenarios using a code similar to the one in the Appendix.

5 Summary

Errors affecting the physical implementation of quantum circuits are not independent, often they are time-correlated. Based on the properties of time-correlated noise, we present an algorithm that allows the correction of one time-correlated error in addition to a new error. The algorithm can be applied to any stabilizer code when the two logical qubits $|0_L\rangle$ and $|1_L\rangle$ are entangled states of 2^n basis states in \mathcal{H}_{2^n} .

The algorithms can be applied to perfect as well as non-perfect quantum codes. The algorithm requires two additional ancilla qubits entangled with the qubit affected by an error during the previous error correction cycle. The alternative is to use a quantum error correcting code capable of correcting two errors in one error correction cycles; this alternative requires a considerably more complex quantum circuit for error correction.

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APPENDIX

```

# Sample code for testing the algorithm
# Requires graphsim from
# \protect\vrule width0pt\protect\href{http://homepage.uibk.ac.at/~sim$705213/work/graphsim.html}{http://homepage.uibk.ac.at/~sim$705213/work/g

import graphsim
import random
random.seed()
gr=graphsim.GraphRegister(22,random.randint(0,1E6))

# Prepare Steane |0> codeword on qubit 0-6
gr.hadamard (4)
gr.hadamard (5)
gr.hadamard (6)
gr.cnot (6, 3)
gr.cnot (6, 1)
gr.cnot (6, 0)
gr.cnot (5, 3)
gr.cnot (5, 2)
gr.cnot (5, 0)
gr.cnot (4, 3)
gr.cnot (4, 2)
gr.cnot (4, 1)

# Entangle qubit 3 with the extra ancilla
gr.cnot(2,7)
gr.hadamard(2)
gr.cnot(2,8)
gr.hadamard(2)

# Errors happen on qubit ?
gr.bitflip(2)
gr.phaseflip(3)

##### Begin current error correction cycle #####

# "Extended" Syndrome extraction
for i in range(9,15):
    gr.hadamard(i)

# Z syndrome
gr.cnot(9,3)
gr.cnot(9,4)
gr.cnot(9,5)
gr.cnot(9,6)

gr.cnot(10,1)
gr.cnot(10,2)
gr.cnot(10,5)
gr.cnot(10,6)
gr.cnot(10,7)

gr.cnot(11,0)
gr.cnot(11,2)
gr.cnot(11,4)
gr.cnot(11,6)
gr.cnot(11,7)

# X syndrome
gr.cphase(12,3)
gr.cphase(12,4)
gr.cphase(12,5)
gr.cphase(12,6)

gr.cphase(13,1)
gr.cphase(13,2)
gr.cphase(13,5)
gr.cphase(13,6)
gr.cnot(13,8)

gr.cphase(14,0)
gr.cphase(14,2)
gr.cphase(14,4)
gr.cphase(14,6)
gr.cnot(14,8)

# "Disentangle" the extra ancilla
gr.hadamard(2)
gr.cnot(2,5)
gr.hadamard(2)
gr.cnot(2,7)

# Measure and output the extra ancilla
print gr.measure(7)
print gr.measure(8)

#gr.print_adj_list ()
#gr.print_stabilizer ()
print

# Measure and output syndrome
for i in range(9,15):
    gr.hadamard(i)
    print gr.measure(i)
print

##### error correction #####

# Error correction according to the syndrome and extra ancilla
gr.bitflip(2)
gr.phaseflip(3)

##### Normal syndrome extraction to double-check #####

for i in range(9+6,15+6):
    gr.hadamard(i)

# Z syndrome
gr.cnot(9+6,3)
gr.cnot(9+6,4)
gr.cnot(9+6,5)
gr.cnot(9+6,6)

gr.cnot(10+6,1)
gr.cnot(10+6,2)
gr.cnot(10+6,5)
gr.cnot(10+6,6)

gr.cnot(11+6,0)
gr.cnot(11+6,2)
gr.cnot(11+6,4)
gr.cnot(11+6,6)

# X syndrome
gr.cphase(12+6,3)
gr.cphase(12+6,4)
gr.cphase(12+6,5)
gr.cphase(12+6,6)

gr.cphase(13+6,1)
gr.cphase(13+6,2)
gr.cphase(13+6,5)
gr.cphase(13+6,6)

gr.cphase(14+6,0)
gr.cphase(14+6,2)
gr.cphase(14+6,4)
gr.cphase(14+6,6)

# Measure and output the syndrome
for i in range(9+6,15+6):
    gr.hadamard(i)
    print gr.measure(i)
print

```